Bounding the complexity of small actions on $\mathbb{R}$-trees.

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[1]

Abstract

We bound the number of orbits of strongly reduced branch points and the number of orbits directions from a branch point in any small stable action of a finitely presented group on an $\mathbb{R}$-tree.

Consider an $\mathbb{R}$-tree T with a minimal small stable isometric action of a finitely presented group Γ. The goal of the paper is to bound the number of orbits of branch points of T and the number of orbits of directions from a branch point of T by a constant $\gamma'(\Gamma)$ depending only on Γ. See section 1 for definitions.

Let's be more precise. The *complexity* $C(x)$ of a point $x \in T$ is the number of orbits of directions at x under $\operatorname{Stab} x$.

Definition. *A point $x \in T$ is* reduced *if $C(x) \neq 2$, or $C(x) = 2$ but no direction at x is fixed by $\operatorname{Stab} x$. We denote by $\operatorname{Red}(T)$ the set of reduced vertices of T.*

Definition. *A point $x \in T$ is* strongly reduced *if one of the following holds:*

1. *$C(x) \geq 3$*
2. *$C(x) = 2$ and no direction at x is fixed by $\operatorname{Stab} x$*
3. *$C(x) = 1$ and $\operatorname{Stab} x$ contains a finitely generated subgroup which fixes no direction at x.*

We denote by $\operatorname{Red}'(T)$ the set of strongly reduced vertices of T.

Remark. If (T, Γ) is minimal, and if a point x has only one orbit of directions in T, then the whole $\operatorname{Stab}(x)$ cannot fix any direction: otherwise there would be exactly one direction at x and $T \setminus \Gamma.x$ would be an invariant subtree. Thus if x is reduced but not strongly reduced if and only if $C(x) = 1$, $\operatorname{Stab} x$ fixes no direction at x but every finitely generated subgroup of $\operatorname{Stab} x$ does.

In small actions of hyperbolic groups, every reduced point is strongly reduced (see section 1.1).

Of course, two points in the same Γ-orbit have the same complexity and are reduced (resp. strongly reduced) simultaneously. Thus, we can talk of the complexity of an orbit of reduced points $\bar{x} = \Gamma.x$.

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Definition. *The* reduced complexity $C(T)$ *of* T *is the sum on each orbit of reduced vertices* $\Gamma.x$ *of the number of orbits of directions at* x*:*

$$C(T) = \sum_{\bar{x} \in \mathrm{Red}(T)/\Gamma} C(\bar{x}).$$

Similarly, we define the strongly reduced complexity $C'(T)$ *by*

$$C'(T) = \sum_{\bar{x} \in \mathrm{Red}'(T)/\Gamma} C(\bar{x}).$$

Clearly, bounding the (strongly) reduced complexity bounds the number of orbits of directions from a branch point and the number of orbits of (strongly) reduced branch points.

Several definitions of smallness appear in the literature. Say that a group H is *algebraically small* (or simply small) if H does not contain any non-abelian free group of rank 2. Say that a group H acts *hyperbolically* on a simplicial tree T if H contains two hyperbolic elements α, β whose axes intersect in a compact set. Say that a group H is *treely small* if H cannot act hyperbolically on any simplicial tree. Algebraically small implies treely small but the converse is not true: $\mathrm{SL}_3\mathbb{Z}$ has property FA (i. e. fixes a point when acting on any simplicial tree) but contains a free group of rank 2. In particular, the class of treely small groups is not closed under taking subgroups. For a subgroup H of a group Γ, this condition can be relaxed in the following way: H is *treely small in* Γ if for any action of Γ on a simplicial tree, H cannot act hyperbolically. Of course, treely small implies treely small in any group. But if $H_1 \subset H_2 \subset \Gamma$, and if H_2 is treely small in Γ, then so is H_1. Moreover, the class of treely small subgroups of Γ is stable under treely small extension: if $H \lhd \hat{H} \subset \Gamma$ with H treely small in Γ, and $\hat{H}/H$ treely small, then $\hat{H}$ is treely small in Γ.

Theorem, Version 1. *Let* Γ *be a finitely presented group. Then there exists an integer* $\gamma'(\Gamma)$ *with the following property.*

Consider a minimal stable action of Γ *on an* $\mathbb{R}$*-tree* T *whose arc stabilizers are treely small in* Γ*. Then the strongly reduced complexity satisfies* $C'(T) < \gamma'(\Gamma)$*.*

Our theorem is based on accessibility of finitely presented groups. Consider Γ a finitely presented group. Consider a class $\mathcal{C}$ of subgroups of Γ which is stable under taking subgroups and under cyclic extension (i. e. if $S \in \mathcal{C}$ and if $\hat{S} \subset \Gamma$ contains S as a normal subgroup with $\hat{S}/S$ cyclic, then $\hat{S} \in \mathcal{C}$). Say that Γ satisfies the *accessibility condition* with respect to $\mathcal{C}$ if there exists a constant $\gamma(\Gamma, \mathcal{C})$ such that every minimal action of Γ on a simplicial tree T with edge stabilizers in $\mathcal{C}$ has a reduced complexity $C(T)$ bounded by $\gamma(\Gamma, \mathcal{C})$.

In [BF91a], Bestvina and Feighn proved accessibility with respect to small subgroups for finitely presented groups.

Accessibility Theorem ([BF91a]). *Let* Γ *be a finitely presented group and* $\mathcal{C}$ *the class of subgroups which are treely small in* Γ*. Then* Γ *satisfies the accessibility condition with respect to* $\mathcal{C}$*.*

Historically, the first accessibility result is Grushko Theorem [Gru40] claiming that $n(A * B) = n(A) + n(B)$ where $n(\Gamma)$ is the minimum cardinal of a generating set of Γ. It follows that every minimal action of a finitely generated group Γ on a simplicial tree with trivial edge stabilizer has complexity at most $6n(\Gamma)-6$.

Then Dunwoody proved that a finitely presented group satisfies the accessibility condition with respect to the class of its finite groups [Dun85]. Linnell proved that the accessibility condition holds for finitely generated groups with respect to the class of its finite subgroups of order bounded by some positive number M (he actually proves something more precise, see [Lin83]).

On the other hand, there exists finitely generated groups which do not satisfy the accessibility condition with respect to the class of their small subgroups ([BF91b]) and even to the class of their finite subgroups ([Dun93]).

Concerning group actions on $\mathbb{R}$-trees, Jiang proved that in a free action of a finitely generated group on an $\mathbb{R}$-tree, there are finitely many orbits of branch points ([Jia93]). Levitt and Paulin proved that any geometric action of a finitely presented group with trivial arc stabilizer has only finitely many orbits of branch points (but this number is not bounded only in terms of Γ) [LP97]. And Gaboriau and Levitt prove that the complexity of very small action of the free group F_n on $\mathbb{R}$-trees is bounded by $6n-6$ [GL95].

Our theorem bounds the complexity of group actions on $\mathbb{R}$-trees under an accessibility hypothesis.

Theorem, Version 2. *Let Γ be a finitely presented group, and $\mathcal{C}$ a class of subgroups of Γ stable under taking subgroups and under cyclic extension. Assume that Γ satisfies the accessibility condition with respect to $\mathcal{C}$ with constant $\gamma=\gamma(\Gamma,\mathcal{C})$.*

Then there exists a constant $\gamma'=\gamma'(\Gamma,\mathcal{C})$ such that for every minimal stable action of Γ on an $\mathbb{R}$-tree T with arc stabilizers in $\mathcal{C}$, the strongly reduced complexity satisfies $C'(T)<\gamma'$.

We still can be more precise. Let $\mathcal{H}=\{H_1,\dots,H_p\}$ be a finite collection of finitely generated subgroups of Γ. We say that Γ is finitely presented rel $\mathcal{H}$ if H_i has a presentation $\langle S_i|R_i\rangle$ with $\#S_i<\infty$, and Γ has a presentation of the form $\langle S\amalg S_1\amalg\dots\amalg S_p|R\amalg R_1\amalg\dots\amalg R_p\rangle$ with $\#S<\infty$ and $\#R<\infty$. We also say that $(\Gamma,\mathcal{H})$ is a finitely presented pair. An action of the pair $(\Gamma,\mathcal{H})$ on an $\mathbb{R}$-tree T is an action of Γ in which every $H_i\in\mathcal{H}$ fixes a point of T. We denote by $(T,\Gamma,\mathcal{H})$ an action of the pair $(\Gamma,\mathcal{H})$ on T. Say that $C<\Gamma$ is treely small in $(\Gamma,\mathcal{H})$ if it cannot act hyperbolically in any action of the pair $(\Gamma,\mathcal{H})$. Say that the pair $(\Gamma,\mathcal{H})$ satisfies the *accessibility condition* with respect to $\mathcal{C}$ if there exists a constant $\gamma(\Gamma,\mathcal{H},\mathcal{C})$ such that every minimal action of the pair $(\Gamma,\mathcal{H})$ on a simplicial tree T with edge stabilizers in $\mathcal{C}$ has a reduced complexity bounded by $\gamma(\Gamma,\mathcal{H},\mathcal{C})$.

It is worth noticing that the result of [BF91a] extends easily to the following result (see section 1.2):

Relative accessibility result ([BF91a]). *Let $(\Gamma,\mathcal{H})$ be a finitely presented pair. Let $\mathcal{C}$ be the class of treely small subgroups in $(\Gamma,\mathcal{H})$.*

Then $(\Gamma,\mathcal{H})$ satisfies the accessibility condition with respect to $\mathcal{C}$.

Theorem, Version 3. *Let $(\Gamma,\mathcal{H})$ be a finitely presented pair. Let $\mathcal{C}$ be a class of subgroups of Γ stable under taking subgroups and under cyclic extension. Assume that $(\Gamma,\mathcal{H})$ satisfies the accessibility condition with respect to $\mathcal{C}$ with constant $\gamma=\gamma(\Gamma,\mathcal{H},\mathcal{C})$.*

Then there exists a constant $\gamma'(\Gamma,\mathcal{H},\mathcal{C})=15\gamma+\gamma 2^{\gamma+\dim H_1(\Gamma;\mathbb{Z}/2)-1}$ such that for every minimal stable action of the pair $(\Gamma,\mathcal{H})$ on an $\mathbb{R}$-tree T with arc stabilizers in $\mathcal{C}$, the strongly reduced complexity satisfies $C'(T)<\gamma'(\Gamma,\mathcal{H},\mathcal{C})$

A group is slender if all its subgroups are finitely generated. Equivalently, a group is slender if any action of any of its subgroups on a simplicial tree fixes a point or a preserves line ([DS99]). Note that an action with finitely generated stabilizers of directions is stable.

Corollary. *Let Γ be a finitely presented group. There exists a constant $\gamma'(\Gamma)$ such that for every minimal action of Γ on an $\mathbb{R}$-tree T with slender direction stabilizer, the strongly reduced complexity satisfies $C'(T) < \gamma'(\Gamma)$.*

For a small action of a hyperbolic group, we have $C(T) = C'(T)$.

Corollary. *Let Γ be a hyperbolic group. There exists a constant $\gamma'(\Gamma)$ such that for every minimal small action of Γ on an $\mathbb{R}$-tree T the reduced complexity satisfies $C(T) < \gamma'(\Gamma)$.*

This corollary is used in [LL] to prove that for any outer automorphism Φ of a non-elementary hyperbolic group, there are infinitely many classes of isogredience classes representing Φ.

The paper is organized as follows. Definitions are recalled in section 1. Section 2 deals with the complexity of graphs of actions. Section 3 shows how to reduce to the case of an almost geometric action which splits as a graph of actions with vertex actions dual to measured foliations on 2-complexes. Section 4 shows how to read the complexity of a foliated 2-complex. Sections 5, 6 and 7 bound the complexity of vertex actions of the three possible type: exotic, surface, and homogeneous. Arguments are put together in section 8 to conclude.

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1 Definitions and preliminaries

In all the sequel, Γ will be a finitely generated group, $\mathcal{H}$ a finite collection of finitely generated subgroups of Γ so that Γ is finitely presented rel $\mathcal{H}$. $\mathcal{C}$ will denote a class of subgroups of Γ satisfying the accessibility condition rel $\mathcal{H}$, stable by taking subgroups and by cyclic extension.

1.1 $\mathbb{R}$-trees, directions, branch points

An $\mathbb{R}$-tree T is an arcwise connected metric space in which any topological arc is isometric to an interval of $\mathbb{R}$. Simplicial trees endowed with a path metric provide examples of $\mathbb{R}$-trees. Let (T, Γ) be a group action on an $\mathbb{R}$-tree (this notation is useful when several groups act on the same tree). We will only consider actions by isometries. This action is *minimal* if every non-empty Γ-invariant subtree of T is T itself (we allow the trivial action of Γ on a point as a minimal action). A *morphism of* $\mathbb{R}$*-trees* is a continuous map $f : T \to T'$ such that every compact interval of T can be subdivided into finitely many sub-arcs on which f is isometric. Given (T_1, Γ_1), (T_2, Γ_2) and a morphism $\varphi : \Gamma_1 \to \Gamma_2$, a morphisms of $\mathbb{R}$-trees $f : T_1 \to T_2$ is φ*-equivariant* (or simply *equivariant*) if $f(\gamma.x) = \varphi(\gamma).f(x)$. All the morphisms we consider are equivariant.

A *direction* at $x \in T$ is a connected component of $T \setminus \{x\}$. Thus, two points y_1, y_2 are in the same direction from x if $[x, y_1] \cap [x, y_2]$ contains more than one point. A *vertex* of T is either

- *a branch point,* i. e. a point $x \in T$ from which there are at least 3 directions,
- a *flip point,* i. e. a point x with exactly two directions and such that there exists $\gamma \in \operatorname{Stab} x$ exchanging the two directions
- or a *terminal point,* i. e. a point $x \in T$ from which there is just one direction.

Note that in a minimal action of a finitely generated group, there are no terminal points, and there are at most countably many vertices. A flip point is always reduced. A reduced point is always a vertex.

Lemma 1.1. *If arc stabilizers are finitely generated and satisfy the ascending chain condition, then any reduced point is strongly reduced.*

Remark. Small actions of hyperbolic groups satisfy the hypotheses of this lemma.

Proof. Because arc stabilizers satisfy the ascending chain condition, every direction stabilizer fix a non-degenerate arc.

Take x a point with $C(x) = 1$. Let H_0 be the stabilizer of a direction η_0 at x. By minimality, there exists $\gamma \in \operatorname{Stab} x$ such that $\gamma.\eta \neq \eta$. If x is not strongly reduced, then the finitely generated group $< \gamma, H_0 > \subset \operatorname{Stab} x$ fixes a direction η_1 at x. The group $H_1 = \operatorname{Stab} \eta_1$ strictly contains H_0. By induction, we construct an infinite ascending chain of arc stabilizers giving a contradiction. □

1.2 Reduced complexity

The original statement of [BF91a] is the following:

Theorem ([BF91a]). *Let Γ be a finitely presented group. Then there exists a constant $\gamma(\Gamma)$ such that the following holds.*

Consider a minimal action of Γ on a simplicial tree with treely small edge stabilizers such that every vertex is reduced. Then T has at most $\gamma(G)$ orbits of vertices.

The following remark is due to G. Levitt. We assume there is no vertex (in the simplicial sense) in T with exactly 2 incident edges which is not a flip point. Consider a non-reduced vertex v. Then there is exactly one edge e incident on v with same stabilizer as v. We say that this edge is non-reduced. Consider the tree T' obtained by collapsing every non-reduced edge in T. Then T' has same reduced complexity as T and every vertex of T' is reduced. This allows to deduce the accessibility theorem given in the introduction.

Let's give a check of the proof the relative version of the accessibility result. A Dunwoody resolution of a minimal simplicial action (T, Γ) is simplicial action (S, Γ) with an equivariant map $f : S_0 \to T$ which embeds every edge and which is simplicial with respect to some equivariant subdivision of S. The main result of [BF91a] may be restated this way:

Assume that (T, Γ) is a minimal simplicial action of a finitely generated group. Assume that T has a Dunwoody resolution (S, Γ) with at most n orbits of vertices. Assume that edge stabilizers of T don't act hyperbolically on S.

Then the reduced complexity satisfies $C(T) \leq 188n + 468\beta_1(\Gamma) + 12 \dim H^1(\Gamma; \mathbb{Z}/2) - 280$.

For a finitely presented group, the existence of a bound $\delta(\Gamma)$ such that every minimal simplicial action (T, Γ) has a Dunwoody resolution with at most $\delta(\Gamma)$ orbits of vertices is proved in [Dun85]. This immediately extends to actions of a finitely presented pair: Γ is finitely presented rel $\mathcal{H} = \{H_1, \dots, H_p\}$ if it acts freely on a simply connected 2-complex X containing simply connected sub-complexes X_i whose stabilizer is H_i, whose orbits are disjoint, and such that $[X \setminus (\cup_i X_i^{(2)})]/\Gamma$ is finite. We first construct an equivariant map $f : X \to T$: send X_i to a point fixed by H_i, and extend it equivariantly to $X^{(0)}$. Extend f linearly on edges, and extend f to $X^{(2)}$ so that for each closed 2-simplex σ, either f is constant on σ or its level sets are the leaves of a foliation which is either regular or has a single 3-pronged singularity in its interior.

We consider the induced measured foliation on X (where simplices on which f is constant are understood as being contained in a leaf). X being simply connected, the space of leaves of X is a simplicial tree S and $f : X \to T$ induces a Dunwoody resolution. The preimage of a vertex of S contains at least a vertex of X or a singularity in a 2-dimensional simplex. Thus, the number of vertices of S is bounded by $\#\left(X^{(0)}/\Gamma\right) + \#\left[\left(X^{(2)} \setminus (\bigcup_i X_i^{(2)})\right)/\Gamma\right]$. Therefore, the accessibility result for relatively finitely presented groups holds.

1.3 Stable actions.

An interval (or a subtree) $I \subset T$ is *non-degenerate* if it contains more than a point. A non-degenerate interval I is *stable* if for every non-degenerate subinterval $J \subset I$, one has equality between pointwise stabilizers $\operatorname{Stab} I = \operatorname{Stab} J$. The action (T, Γ) is *stable* if every non-degenerate interval in T contains a stable interval. For example, if the set of arc stabilizers satisfy the ascending chain condition, then the action is stable. In this case the stabilizer of a direction fixes a non-degenerate arc.

2 Complexity of a graph of actions

2.1 Graph of actions

Definition. *A* graph of actions *on $\mathbb{R}$-trees $\mathcal{G}$ is a graph of groups with vertex groups Γ_v, edge groups Γ_e and edge morphisms $i_e : \Gamma_e \to \Gamma_{t(e)}$ together with the following data:*

- *for each vertex v, an action (T_v, Γ_v) of the corresponding vertex group*
- *for each oriented edge e incident to $v = t(e)$, an attaching point $p_e \in T_v$ fixed by $i_e(\Gamma_e)$*
- *a (maybe 0) length for each non-oriented edge of Q.*

We define the fundamental group of $\mathcal{G}$ to be the fundamental group of the underlying graph of groups. To a graph of actions $\mathcal{G}$ naturally corresponds an action $(T_{\mathcal{G}}, \pi_1(\mathcal{G}))$: it is obtained from the universal cover $\tilde{\mathcal{G}}$ of the graph of groups underlying $\mathcal{G}$ by replacing a vertex $\tilde{v}$ of $\tilde{\mathcal{G}}$ by a copy of the corresponding vertex $\mathbb{R}$-tree T_v, by gluing equivariantly edges incident to a vertex v on T_v according to the attaching points, and by collapsing the 0-length edges (see for instance [Lev94, Gui98]).

We say that an action (T, Γ) splits as a graph of actions if (T, Γ) is equivariantly isometric to the action $(T_{\mathcal{G}}, \pi_1(\mathcal{G}))$ corresponding to a graph of actions $\mathcal{G}$. If $T = T_{\mathcal{G}}$ is a splitting of T as a graph of actions, then consider the equivariant family $\mathcal{S}$ of non-degenerate subtrees of T corresponding either to a non-degenerate vertex tree of $\mathcal{G}$ or to an edge with positive length. Then $\mathcal{S}$ covers T and for $S, S' \in \mathcal{S}$, $S \cap S'$ contains at most one point. Moreover, every arc in T is covered by finitely many elements of $\mathcal{S}$. There is a reciprocal to this observation:

Lemma 2.1. *Let (T, Γ) be an action of a group on an $\mathbb{R}$-tree. Assume that T is covered by a family of closed non-degenerate subtrees $\mathcal{S}$ invariant under the action of Γ such that*

- *for $S \neq S' \in \mathcal{S}$, $S \cap S'$ contains at most one point*
- *every arc in T is covered by finitely many elements of $\mathcal{S}$.*

Then T has a natural decomposition into a graph of actions.

Remark. The graph of actions obtained satisfies that

- the vertex action of an endpoint of an edge of positive length is a point
- edges of length 0 have exactly one endpoint whose vertex action is a point
- If e_1, e_2 are two different length-0 edges incident on the same vertex v such that T_v is non-degenerate then p_{e_1} and p_{e_2} are not in the same orbit in (T_v, Γ_v)

We say that a graph of actions obtained by this lemma is a graph of actions in standard form.

Proof. We say that an arc in an $\mathbb{R}$-tree a *simplicial arc* if it doesn't meet any branch point of T except maybe at its endpoints. Let E_1 be the subset of $\mathcal{S}$ consisting of simplicial arcs which are not flipped by any element of Γ. Let $V_1 = \mathcal{S} \setminus E_T$. Let $V_0 = \{x \in T | \exists S \neq S' \in \mathcal{S} \text{ s.t. } x \in S \cap S'\}$ and let $E_0 = \{(x, S) \in V_0 \times V_1 | x \in S\}$. If (T, Γ) is not minimal, we may need to add to V_0 some endpoints of simplicial arcs in E_1 if they are have valence 1 in T. $V = V_1 \cup V_0$ and $E = E_1 \cup E_0$ are the sets of vertices and non-oriented edges of a graph $\tilde{\mathcal{G}}$: the endpoints of $e = (x, S) \in E_0$ are $x \in V_0$ and $S \in V_1$, and the endpoints of $S = [a, b] \in E_1$ are the points $a, b \in V_0$. We assign the length 0 to any edge in E_0/Γ, and we assign the positive length of the corresponding simplicial arc in $\mathcal{S}$ to any edge in E_1/Γ. Clearly, $\tilde{\mathcal{G}}$ is a tree endowed with an isometric action of Γ. Therefore, we naturally get a graph of actions whose graph of groups is $\tilde{\mathcal{G}}/\Gamma$, whose vertex actions are conjugacy classes of the actions $(S, \mathrm{Stab}\,(S))$ for $S \in V$. $\square$

2.2 The complexity of a graph of actions

Lemma 2.2. *Consider a graph of actions on $\mathbb{R}$-trees $\mathcal{G}$ in standard form, and assume that every vertex action is minimal.*

Let A be the set of oriented (length-0) edges $e \in E(\mathcal{G})$ whose attaching point p_e is not reduced in the corresponding vertex tree.

Let B_1 be the set of vertices $v \in V_0(\mathcal{G})$ of valence 2 in $\mathcal{G}$ such that, both incident edges have positive length and at least one of the edge morphisms is onto.

Let B_2 be the set of vertices $v \in V_0(\mathcal{G})$ of valence 2 in $\mathcal{G}$ such that, some edge e incident on v has positive length and i_e is onto, and the other edge e' incident on v has length 0, and $p_{\bar{e}'}$ has exactly one orbit of directions in $T_{t(\bar{e}')}$.

Let $a = \#A$, $b_1 = \#B_1$ and $b_2 = \#B_2$. Then

$$C(T_{\mathcal{G}}) = \sum_v C(T_v) + 2\#\{\text{non-oriented edges of positive length}\} + 2a - 2(b_1 + b_2).$$

Equivalently, if $\mathcal{G}_0$ denotes the graph of groups obtained from $\mathcal{G}$ by collapsing length-0 edges and $\tilde{\mathcal{G}}_0$ its Bass-Serre universal cover,

$$C(T_{\mathcal{G}}) = C(\tilde{\mathcal{G}}_0) + \sum_v C(T_v) + 2a - 2b_2.$$

Remark. If vertex actions are not minimal then an additional term has to be included because some other vertices in $V_0(\mathcal{G})$ may become non-reduced in $T_{\mathcal{G}}$.

Proof. Consider $(T_{\mathcal{G}}, \Gamma)$ the action corresponding to the graph of actions $\mathcal{G}$. Let $\mathcal{S}$ be the collection of non-degenerate subtrees of $T_{\mathcal{G}}$ corresponding the decomposition of $T_{\mathcal{G}}$ into a graph of actions. For $x \in T_{\mathcal{G}}$, let $\mathcal{S}_x$ be the set of $S \in \mathcal{S}$ containing x. For $S \in \mathcal{S}_x$, let Γ_S denote the setwise stabilizer of S, and let $C_S(x)$ be the number of orbits of directions of x in (S, Γ_S). Since any arc $[x, x']$ is covered by finitely many elements of $\mathcal{S}$, any direction at $x \in T_{\mathcal{G}}$ corresponds to a direction at x in some $S \in \mathcal{S}_x$. Thus,

$$C(x) = \sum_{S \in \mathcal{S}_x / \mathrm{Stab}\,(x)} C_S(x).$$

Hence,

$$C(T_{\mathcal{G}}) = \sum_{x \in \mathrm{Red}(T_{\mathcal{G}})/\Gamma} C(x) = \sum_{x \in \mathrm{Red}(T_{\mathcal{G}})/\Gamma} \; \sum_{S \in \mathcal{S}_x / \mathrm{Stab}\,(x)} C_S(x).$$

Therefore, the complexities $C(S)$ for $S \in \mathcal{S}/\Gamma$ add up to the complexity of $T(\mathcal{G})$ except that a point x which is reduced in $T_{\mathcal{G}}$ might not be reduced in T_S or vice-versa. This translates into the following equality:

$$C(T_{\mathcal{G}}) = \sum_{S \in \mathcal{S}/\Gamma} C(S) + \sum_{\{(x,S) \in E_0(T_{\mathcal{G}})/\Gamma | x \notin \mathrm{Red}(S)\}} C_S(x) - \sum_{\{x \in V_0(T)/\Gamma | x \notin \mathrm{Red}(T_{\mathcal{G}})\}} C(x)$$

$$= \sum_{S \in \mathcal{S}/\Gamma} C(S) + 2\#\big\{(x,S) \in E_0(T_{\mathcal{G}})/\Gamma | x \notin \mathrm{Red}(S)\big\} - 2\#\big\{x \in V_0(T)/\Gamma | x \notin \mathrm{Red}(T_{\mathcal{G}})\big\}$$

Since $a = \#\big\{(x,S) \in E_0(T_{\mathcal{G}})/\Gamma | x \notin \mathrm{Red}(S)\big\}$, we have to prove that $x \in V_0(T)$ is not reduced in $T_{\mathcal{G}}$ if and only if the corresponding vertex $v \in \mathcal{G}$ lies in $B_1 \cup B_2$.

A point $x \in V_0(T_{\mathcal{G}})$ is not reduced in $T_{\mathcal{G}}$ if and only if $C(x) = 2$ and $\mathrm{Stab}\,(x)$ fixes a direction η_0 in $T_{\mathcal{G}}$. Thus $\mathrm{Stab}\,(x)$ preserves the tree $S_0 \in \mathcal{S}_x$ intersecting this direction. Since $x \in V_0(T_{\mathcal{G}})$, x lies in some $S_1 \in \mathcal{S}_x \setminus \{S_0\}$. Since $C(x) = 2$, $C_{S_1}(x) = C_{S_0}(x) = 1$, and $\mathcal{S}_x = \{S_0, \mathrm{Stab}\,(x).S_1\}$. In particular, v is a valence 2 vertex in $\mathcal{G}$. Moreover, η_0 is the only direction at x in S_0, so S_0 cannot be minimal, and by hypothesis, S_0 is a simplicial arc. It is now clear that if S_1 is a simplicial arc then $v \in B_1$, and $v \in B_2$ otherwise. $\square$

We will use the previous Lemma in the following form:

Corollary 2.3. *Let $\mathcal{G}$ and $\mathcal{G}'$ be 2 graphs of actions on $\mathbb{R}$-trees having the same underlying metric graph of groups G. Assume that every vertex action of $\mathcal{G}$ and $\mathcal{G}'$ is minimal and that any arc stabilizer in (T_v, Γ_v) fixes an arc in (T'_v, Γ_v).*

Then we can move attaching points of $\mathcal{G}'$ so that

$$C(T_{\mathcal{G}}) \leq C(T'_{\mathcal{G}}) + \sum_{v \in V(G)} \big[C(T_v) - C(T'_v)\big] + 2C(\tilde{G}_0)$$

where G_0 is the graph of groups obtained from G by collapsing edges of length 0, and $\tilde{G}_0$ denotes its universal cover.

Proof. Lemma 2.2 implies that

$$C(T_{\mathcal{G}}) = C(T_{\mathcal{G}'}) + \sum_{v \in V(G)} \left[C(T_v) - C(T'_v)\right] + 2(a - a') - 2(b_2 - b'_2).$$

We can manage so that $a' = a$ in the following way: consider an edge $e \in E(G)$ having an attaching point p_e which is not reduced in T_v where $v = t(e)$. The stabilizer of p_e in (T_v, Γ_v) fixes an arc in T_v, so it also fixes an arc I' in T'_v. Thus, we can change the attaching point p'_e of e in T'_v to any point in I', so we can choose an attaching point which is not one of the countably many vertices. This way we get $a = a'$.

Since $B_1 = B'_1$, we just need to prove that $b'_2 \leq C(\tilde{\mathcal{G}}_0)$. Consider $\tilde{\mathcal{G}}$ the universal cover of $\mathcal{G}$. We first prove that if v is any vertex in $\tilde{\mathcal{G}}$ such that T_v is non-degenerate, then its image $\bar{v}$ in $\tilde{\mathcal{G}}_0$ is reduced. Let $x \in T_v$ such that $(x, T_v) \in E_0(T_{\mathcal{G}})$. By minimality of vertex actions, there is an element of Γ_v which does not fix x. This means that Γ_v fixes no edge in $\tilde{\mathcal{G}}$. Since the map from $\tilde{\mathcal{G}}$ to $\tilde{\mathcal{G}}_0$ consists in collapsing length-0 edges, we get that Γ_S fixes no edge in $\mathcal{G}_0$. Therefore, the image of v in $\tilde{\mathcal{G}}_0$ is reduced.

Let $v \in B_2$, let e' be the length-0 edge incident on v and let $v' = t(\bar{e}')$. $T_{v'}$ is minimal and non-degenerate, hence the image of v' is reduced in $\mathcal{G}_0$. Consider the map from B_2 to $E(\mathcal{G}_0)$ sending a vertex $v \in B_2$ to the image in $\mathcal{G}_0$ of the edge positive length incident on v in $\mathcal{G}$. Then its image is contained in the set of oriented edges whose endpoint is reduced. Since this map is one to one, we get that $b'_2 \leq C(\tilde{\mathcal{G}}_0)$. $\square$

3 Strong approximations, finitely presented pairs and almost geometric actions.

The goal of this section is to reduce the proof of the Theorem to actions dual to pure minimal systems of isometries.

3.1 Strong approximations are not less complex.

In this section we prove that if an action $(T, \Gamma, \mathcal{H})$ is a strong limit of a sequence of actions whose reduced complexities are bounded by M, then the strongly reduced complexity of $(T, \Gamma, \mathcal{H})$ is also bounded by M.

Definition ([LP97]). *Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be a sequence of finitely generated groups with epimorphisms $\varphi_{ij} : \Gamma_i \twoheadrightarrow \Gamma_j$ for $i < j$ so that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for $i < j < k$. Let Γ be the direct limit $\varinjlim \Gamma_i$ and $\varphi_i : \Gamma_i \twoheadrightarrow \Gamma$ the natural morphisms.*

A sequence of minimal actions on $\mathbb{R}$-trees (T_i, Γ_i) converges strongly *to (T, Γ) if*

- *for every $i < j$, there are equivariant morphisms of $\mathbb{R}$-trees $f_{ij} : T_i \to T_j$ and $f_i : T_i \to T$*
- *for every $i < j < k$, $f_{jk} \circ f_{ij} = f_{ik}$ and $f_j \circ f_{ij} = f_i$*
- *for every finite subtree[2] K in some T_i, there exists $j > i$ such that f_j is an isometry in restriction to $f_{ij}(K)$.*

[2] a *finite subtree* is the convex hull of finitely many points

Remark. When Γ is finitely presented, the morphisms φ_{ij} and φ_i are isomorphisms for large enough i, so $\Gamma_i = \Gamma$. We will see that an action of a finitely presented pair $(\Gamma, \mathcal{H})$ is a strong limit of (almost-)geometric actions of the pair $(\Gamma, \mathcal{H})$ i. e. $\Gamma_i = \Gamma$, $\varphi_{ij} = \mathrm{Id}$, and elements of $\mathcal{H}$ are elliptic (see lemma 3.3, prop. 3.4).

Proposition 3.1. *Let $(T, \Gamma, \mathcal{H})$ be an action of a finitely presented pair whose* strongly *reduced complexity satisfies $C'(T) \geq N$ for some* finite *number N. Assume that $(T_i, \Gamma, \mathcal{H})$ converges strongly to $(T, \Gamma, \mathcal{H})$. Then for i sufficiently large, the reduced complexity of T_i satisfies $C(T_i) \geq N$.*

Remark. We don't assume *a priori* that the complexity of T is finite but this is a consequence of Theorem 1 when T is a small stable action.

Proof. Consider a sequence $(T_i, \Gamma, \mathcal{H})$ converging strongly to $(T, \Gamma, \mathcal{H})$. Denote by $f_i : T_i \to T$ the corresponding morphisms of $\mathbb{R}$-trees.

It is a standard fact that the definition of strong convergence implies that for any finite tree $K \subset T$ and any finite subset $F \subset \Gamma$, we can isometrically *lift* K and the action of F to some $K_i \subset T_i$ in the following sense: ${f_i}_{|K_i}$ is an isometry between K and K_i, and for every $\gamma \in F$ and every $a, b \in K_i$, $\gamma.a = b$ if and only if $\gamma.f_i(a) = f_i(b)$.

Indeed, take a finite set $\{x_1, \dots, x_p\} \subset T$ whose convex hull is K. Choose a preimage y_k of each x_k in T_1. Let K_1 bet the convex hull of $\{\gamma.y_k \,|\, \gamma \in F, k = 1, \dots, p\}$. Then for i large enough, $K_i = f_{1i}(K_1)$ embeds into T and provides the desired lift.

Claim 3.2. *Consider a strongly reduced vertex $x \in T$, and a* finite *number $n \leq C(x)$. Then for large enough i, there is a preimage x' of x in T_i such that x' is a reduced vertex of T_i and $C(x') \geq n$.*

The claim implies the proposition since if $x_1, \dots, x_p \in T$ denote some strongly reduced vertices in distinct orbits such that $C(x_1) + \cdots + C(x_p) \geq N$ then we find some reduced points $x'_1, \dots, x'_p \in T_i$ for some i such that $C(x'_1) + \cdots + C(x'_p) \geq N$. Moreover, these vertices are in distinct orbits since $f_i(x'_k) = x_k$ and f_i is equivariant.

So let's turn to the proof of the claim. Consider a strongly reduced vertex $x \in T$ and $n \leq C(x)$ (take $n = C(x)$ if $C(x) < \infty$ and $n \geq 3$ if $C(x) = \infty$). Take a finite subtree $K \subset T$ containing $\{x\}$ and at least n directions at x which are not in the same orbit under $\mathrm{Stab}\,(x)$. Take i big enough so that K lifts to some $K_i \subset T_i$ such that f_i restricts to an isometry on K_i and let $x' = ({f_i}_{|K_i})^{-1}(x)$. Thus $C(x') \geq n$. So we just have to check that x' is a reduced vertex for large enough i.

Let's first check that x' is a vertex in T_i. If x is a branch point of T, then $n \geq 3$ and x' is branch point in K_i and therefore in T_i. If x is a flip point with g flipping the two directions at x, we take K containing the two directions at x and i large enough so that g flips the two directions at x' in K_i. If x' is not a branch point in T_i, then it is a flip point and hence a vertex.

Let's prove that x' is reduced. If x has at least 3 orbits of directions, then so does x'. If x has exactly two orbits of directions (say $\mathrm{Stab}\,(x).\eta_1$ and $\mathrm{Stab}\,(x).\eta_2$ with $\eta_1, \eta_2 \subset K$), then there exists $g_1, g_2 \in \mathrm{Stab}\,(x)$ which don't fix η_1 and η_2 respectively. Take i big enough so that g_1 and g_2 fix x'. Since f_i is equivariant, g_1 and g_2 respectively don't fix the preimage under ${f_i}_{|K_i}$ of η_1, η_2 so x' is reduced.

Finally, assume that x has only one orbit of directions. Since x is strongly reduced, there exists a finitely generated group H which fixes no direction at x. We assume that i

large enough so that H fixes x'. If x' is not reduced, then $\operatorname{Stab} x'$ fixes a direction at x', and so does H. Hence H fixes a direction at x, a contradiction.

The conclusion is that x' is reduced for large enough i, and $C(x') \geq N$. $\square$

3.2 Strong approximation by an almost geometric action.

We restate here the fact any action has a strong approximation by a geometric action and that a geometric action splits into pure components (see [Gui98, prop. 4.1] or [BF95], [GLP94]).

Foliated 2-complexes, geometric actions, systems of isometries.

Consider a foliated 2-complex Σ (see [LP97]). Roughly speaking, a foliated 2-complex is a 2-dimensional simplicial complex where 2-simplices are endowed with a regular measured foliation, and 1-simplices are assigned a measure the set of which is invariant under holonomy.

Edges may be transverse to the foliation or contained in a leaf. Given a finitely generated group Γ and a morphism $\rho : \pi_1\Sigma \to \Gamma$, consider the corresponding Galois covering $\bar{\Sigma}$ of Σ. The transverse measure of the lifted foliation on $\bar{\Sigma}$ induces a pseudo-metric on $\bar{\Sigma}$. The space $T_{\bar{\Sigma}}$ obtained by making this pseudo-metric Hausdorff is called *the leaf space made Hausdorff of* $\bar{\Sigma}$. We say that leaf space of $\bar{\Sigma}$ is Hausdorff to mean that two different leaves are at a non-zero distance for this pseudo-metric. When $\ker \rho$ is normally generated by free homotopy classes of loops contained in leaves of Σ, $T_{\bar{\Sigma}}$ is an $\mathbb{R}$-tree ([Lev93a, LP97]).

Definition. *When every component of the preimage $\bar{D}$ of D in $\bar{\Sigma}$ isometrically embeds into $T_{\bar{\Sigma}}$, we say that $\bar{\Sigma}$ is* tame.

Definition. *An action of a finitely generated group (T, Γ) is* geometric *if it is isomorphic to some $(T_{\bar{\Sigma}}, \Gamma)$ where $\bar{\Sigma}$ is tame (the leaf space of $\bar{\Sigma}$ is not required to be Hausdorff) (see [LP97]). We sometimes say that (T, Γ) is* dual *to Σ.*

Many foliated 2-complexes are obtained as the suspension of a system of isometries. A system of isometries X on a metric graph D is a finite set of partially defined isometries between non-empty compact connected subsets of D. A partial isometry $\varphi \in X$ is called a *generator*. A generator is a *singleton* if its domain contains exactly one point. When D is a multi-interval (that is a finite union of compact intervals), there is a corresponding *open* system of isometries $\mathring{X}$ which is the restriction of the generators of X to the interior of their domain. We say that X is *pure* if X contains no singleton and if the $\mathring{X}$-orbit of every point in $\mathring{D}$ is dense in D.

The *suspension* Σ of X is the foliated 2-complex obtained by gluing on D, for each generator $\varphi \in X$, a band $(\operatorname{dom}\varphi) \times [0, 1]$ where $(x, 0)$ and $(x, 1)$ are glued with x and $\varphi(x)$ respectively. Each band is foliated by $\{*\} \times [0, 1]$, and we consider the transverse measure which gives to every arc of D a measure equal to its length.

If an action (T, Γ) is geometric, then it is dual to the suspension of a system of isometries induced by the generators of Γ on a finite subtree $K \subset T$ ([LP97]). Moreover, if Γ is finitely presented and if $H'_1, \dots, H'_p < \Gamma$ are finitely generated subgroups which are elliptic in T, then it is dual to a foliated 2-complex Σ where $\pi_1(\bar{\Sigma})$ is normally generated by curves contained in leaves, and each H'_i preserves a leaf in $\bar{\Sigma}$. Such a presentation of the action is said to be in *standard form.*

Strong approximations of actions of a finitely presented pair

We now prove that actions of a finitely presented pair $(\Gamma,\mathcal{H})$ have strong approximations by geometric actions of the pair $(\Gamma,\mathcal{H})$ (i. e. $\Gamma_i=\Gamma$, $\varphi_{ij}=\mathrm{Id}$, and elements of $\mathcal{H}$ are elliptic in T_i).

Lemma 3.3. *Let $(T,\Gamma,\mathcal{H})$ be a minimal action of a finitely presented pair. Then $(T,\Gamma,\mathcal{H})$ is a strong limit of geometric actions of the pair $(\Gamma,\mathcal{H})$.*

Proof. Start with a sequence (T_i,Γ_i) of geometric actions of finitely presented groups converging strongly to (T,Γ). Since $\mathcal{H}$ is made of finitely many finitely generated groups, by lifting the action of its generators to T_i we can assume that for i big enough, there are finitely generated groups $H'_1,\dots,H'_p<\Gamma_i$ such that $\varphi_i(H'_k)=H_k$ and H'_k preserves a leaf in $\bar{\Sigma}_i$.

Consider Σ_i a foliated 2-complex and $\rho_i:\pi_1\Sigma_i\to\Gamma_i$, so that T_i is the leaf space made Hausdorff of the corresponding Galois covering $\bar{\Sigma}_i$. We take this presentation of T_i in standard form so that we can assume that $\pi_1\bar{\Sigma}_i$ is normally generated by curves contained in leaves and H'_k fix a leaf in $\bar{\Sigma}_i$.

Since $(\Gamma,\mathcal{H})$ is finitely presented, $\ker\varphi_i$ is normally generated by subgroups of $H'_1,\dots,H'_p$ for large enough i. Hence, $\bar{\Sigma}_i/\ker\varphi_i$ is normally generated by curves contained in leaves. So, its leaf space made Hausdorff T'_i is an $\mathbb{R}$-tree, endowed with an action of the pair $(\Gamma,\mathcal{H})$. Since the map $f_i:T_i\to T$ factors through T'_i, the sequence of geometric actions $(T'_i,\Gamma,\mathcal{H})$ converges strongly to T_i. $\square$

Almost geometric actions

Definition. *An action of a finitely generated group on an $\mathbb{R}$-tree is* almost geometric *if it splits as a finite graph of actions on $\mathbb{R}$-trees $\mathcal{G}$ such that for every vertex action (T_v,Γ_v), there is a normal subgroup $N_v\lhd\Gamma_v$ contained in the kernel of (T_v,Γ_v) and $(T_v,\Gamma_v/N_v)$ is geometric. We use the notation $\Gamma_v^0=\Gamma_v/N_v$.*

Remark. If the subgroups N_v are trivial, then the action is geometric (see [Gui00]).

An almost geometric action needn't be geometric: a free action of $\mathbb{Z}^n$ on the real line by translations is geometric. Now consider the action of the free group F_n on the real line induced by a morphism $F_n\to\mathbb{Z}^n$. The obtained action of F_n is not geometric since it is proved in [GL95] that in a geometric action of F_n, the set of fixed points of any element is compact.

The following proposition is a restatement of the fact that a system of isometries splits into pure components, and that a minimal component gives an action whose arc stabilizers are contained in its kernel (under a stability hypothesis). It was proved in the setting of finitely presented group in [Gui98] but adapts adapts immediately to finitely presented pairs using 3.3 ([Gui98] Prop. 4.1, Lemma 4.2).

Proposition 3.4 (D. Gaboriau, [Gui98], prop 4.1). *A stable action $(T,\Gamma,\mathcal{H})$ of a finitely presented pair is a strong limit of almost geometric actions $(T_i,\Gamma,\mathcal{H})$ having a nice decomposition in the sense of the definition below.*

Definition 3.5. *An almost geometric action has a* nice *decomposition if it decomposes into a graph of actions $(T_{\mathcal{G}},\Gamma)$ with the following properties (using notations of the previous definition).*

- *Vertex groups Γ_v are finitely generated*

- *Every action (T_v, Γ_v^0) is minimal (it may be degenerate), it has trivial arc stabilizers, and its orbits are dense in every segment.*

- *If T_v is non-degenerate, then the following holds: there is a pure minimal system of isometries X_v, a Γ_v^0-cover $\bar{\Sigma}_v$ of its suspension Σ_v, such that $\pi_1(\bar{\Sigma}_v)$ is normally generated by curves contained in leaves, and (T_v, Γ_v^0) is the leaf space made Hausdorff of $\bar{\Sigma}_v$.*

- *$\bar{\Sigma}_v$ is tame and the leaf space of $\bar{\Sigma}_v$ is Hausdorff.*

In view of Lemma 2.2 and Proposition 3.1 we want to analyze actions coming from pure minimal system of isometries.

3.3 Pruning process and the 3 types of pure systems of isometries

There are 3 exclusive types of pure systems of isometries: homogeneous type, surface type, and exotic type. When the $\overset{\circ}{X}$-orbits are locally the trace of a *group* of isometries of $\mathbb{R}$, then X is said to be *homogeneous* (or axial). This group of isometries is well defined by X up to conjugacy. It is called the *group of periods* of X. An homogeneous system of isometries may either be orientable or non-orientable according to the fact that its group of periods is orientable or not. If X is homogeneous and if $T_{\bar{\Sigma}}$ has trivial arc stabilizers, then $T_{\bar{\Sigma}}$ is a line, and Γ is isomorphic to the group of periods (see [Gui98], [BF95], [Pau97]).

For the two other types of system of isometries we need to recall the definition of the pruning process (or Rips Machine 1). A *Rips move* is an operation on X, D and Σ which doesn't change $T_{\bar{\Sigma}}$, tameness, and the fact that the space of leaves of $\bar{\Sigma}$ is Hausdorff. Thanks to a theorem by Gaboriau, given a non-homogeneous system of isometries, one can perform Rips moves on it so that it generators become *independent*: this means that no word in the $\overset{\circ}{X}$-generators fixes a point in D (see [Gab97, Pau97, Gui98]). A system of isometries with independent generators has the following property: the total length of the domain D equals the sum of the lengths of the domains of the generators ([Lev93b, GLP94]). This means that

- either every point of D but finitely many of them lie in the domain of exactly 2 generators,

- or the set E of points of D which lie in the domain of exactly one generator is non-empty.

In the first case, Σ is a surface with boundary with a measured foliation. In the second case, E is a finite union of intervals; these intervals are open in D and their closures don't intersect by purity of X. So we can *prune* E: we define a new system of isometries whose domain is $D' = D \setminus E$, and the new set of generators consists in the restrictions of generators of X to D'. This pruning operation is a Rips move, generators remain independent, and X' is still pure ([Gab96]). Note that the pruning operation removes all the points in D which are terminal vertices in their leaf.

Therefore, either the suspension of the new system of isometries is a surface, or the pruning operation can be iterated. If this pruning process stops, then the final 2-complex is a surface and we say that X is of *surface type* (or interval exchange). Otherwise, X is called *exotic* (thin in [BF95]).

We say that a pruning operation is an *interior pruning* if E doesn't intersect ∂D. According to [BF95, Gab96], all the prunings but finitely many of them are interior prunings.

4 Reading the complexity from the foliation.

Consider X a system of isometries, Σ it suspension, $\rho : \pi_1\Sigma \twoheadrightarrow \Gamma$, and $T_{\bar{\Sigma}}$ the dual tree. The goal of this section is to compute the complexity of $T_{\bar{\Sigma}}$ in terms of Σ when $\bar{\Sigma}$ is tame, and its leaf space is Hausdorff. This section follows [GL95] which deals with actions of a free group. Notations are those of section 3.2.

Let x be a point in $T_{\bar{\Sigma}}$, $\overline{\mathcal{L}}$ the corresponding leaf in $\bar{\Sigma}$, and $\mathcal{L}$ its projection in Σ. $\mathcal{L}$ and $\overline{\mathcal{L}}$ come with their natural graph structure. Note that $\operatorname{Stab}(x)$ is conjugate to $\rho(i_*(\pi_1(\mathcal{L})))$ where $i : \mathcal{L} \hookrightarrow \Sigma$ denotes the inclusion.

Now consider the *graph of directions* $\mathcal{DL}$ of $\mathcal{L}$: a vertex of $\mathcal{DL}$ is a direction in D from a point $v \in \mathcal{L} \cap D$. Here we think of a direction at v as a germ of isometric map $d : [0, \varepsilon] \to D$ with $d(0) = v$. We put an edge between two directions d, d' at v, v' for every generator φ such that $\varphi \circ d$ is defined on a non-degenerate interval and $d' = \varphi \circ d$. We denote by $q : \mathcal{DL} \to \mathcal{L}$ the natural map sending d to $d(0)$. Similarly, there is a graph of directions $\overline{\mathcal{DL}}$ of $\overline{\mathcal{L}}$: its vertices are directions from a point $v \in \overline{\mathcal{L}}$ in $\bar{D}$, we put an edge between d and d' for each band between $d(0)$ and $d'(0)$ whose holonomy sends d to d'. We denote by $\bar{q}$ the natural map $\overline{\mathcal{DL}} \to \overline{\mathcal{L}}$. When the leaf space of $\bar{\Sigma}$ is Hausdorff, we have $\operatorname{Stab}(\overline{\mathcal{L}}) = \operatorname{Stab}(x)$, and $\mathcal{DL} = \overline{\mathcal{DL}}/\operatorname{Stab}(\overline{\mathcal{L}})$.

The following lemma computes the complexity of $x \in T_{\bar{\Sigma}}$ in terms of the graph of directions at $\mathcal{L}$. This is a direct generalization of Lemma III.5 in [GL95].

Lemma 4.1. *We assume that $\bar{\Sigma}$ is tame, that the leaf space of $\bar{\Sigma}$ is Hausdorff, and that arc stabilizers of $T_{\bar{\Sigma}}$ are trivial.*

- *The set of directions at $x \in T_{\bar{\Sigma}}$ is in one-to-one correspondence with the set of connected components of $\overline{\mathcal{DL}}$.*
- *The set of orbits of directions at x is in one-to-one correspondence with the set of connected components of $\mathcal{DL}$.*

Since a complexity of a point an hence of the action can be read off Σ we will denote by $C(p)$ the number of components of $\mathcal{DL}$, the graph of directions at a point $p \in \Sigma$. We will also use the abuse of notation $C(\Sigma) = C(T_{\bar{\Sigma}})$.

Corollary 4.2. *If $\mathcal{L}$ is a regular leaf, then x is not a branch point. In this case, x is a flip point if and only if $\mathcal{L}$ is transversally non-orientable.*

Proof of the corollary. If $\mathcal{L}$ is regular, then $\overline{\mathcal{DL}}$ is a 2-covering of $\overline{\mathcal{L}}$. Moreover, $\overline{\mathcal{DL}}$ is not connected because $\bar{\Sigma}$ is tame. Thus $\mathcal{DL}$ is a the disjoint union of 2 copies of $\mathcal{L}$, so x is not a branch point by the lemma. Now x is a flip point if and only if an element of $\operatorname{Stab}(x)$ exchanges those two copies. This means that $\mathcal{L}$ is transversally non-orientable. ☐

Proof of the lemma. Since $\bar{\Sigma}$ is tame, there is a natural map Δ from $\overline{\mathcal{DL}}$ to the set of directions at x in $T_{\bar{\Sigma}}$: just project on $T_{\bar{\Sigma}}$ a germ $\bar{d} : [0, \varepsilon] \to \bar{D}$. Lemma 4.3 will say that Δ is onto.

If $\bar{d}, \bar{d}'$ are in the same component of $\overline{\mathcal{DL}}$, then there is a segment in $\mathcal{L}$ joining $q(\bar{d})$ to $q(\bar{d}')$ whose holonomy sends d to $\bar{d}'$ so that $\Delta(\bar{d})$ and $\Delta(\bar{d}')$ define the same direction in $T_{\bar{\Sigma}}$.

Conversely, if $\Delta(\bar{d}) = \Delta(\bar{d}')$ then for η small enough, $\forall t \in [0,\eta]$, $\bar{d}(t)$ and $\bar{d}'(t)$ are in the same leaf. By segment-closed property (see [GLP95, Th.2.3]), there exists $\eta' < \eta$ and a word w in the generators of X, such that the projections d, d' of $\bar{d}, \bar{d}'$ in Σ are such that $w \circ d'(t) = d(t)$. Thus w lifts to a band of leaves whose holonomy sends $\bar{d}$ to some $\gamma.\bar{d}'$. Therefore, for $t < \eta'$, $\gamma.\bar{d}'(t)$ and $\bar{d}'(t)$ are in the same leaf. This means that γ fixes an arc in $T_{\bar{\Sigma}}$ so $\gamma = 1$. Thus the band of leaves defined by w provides a path in $\mathcal{DL}$ joining $\bar{d}$ to $\bar{d}'$. This proves the first part of the lemma. The second part is clear since Δ is Stab (x)-equivariant. $\square$

Lemma 4.3. *Any germ of isometric map $d : [0,\varepsilon] \to T_{\bar{\Sigma}}$ can be lifted to $\bar{D}$.*

Remark. In this lemma, the fact that the leaf space of $\bar{\Sigma}$ is Hausdorff is not necessary.

Proof. Consider two preimages a, b of $d(0), d(\varepsilon)$ in $\bar{D}$. Take a path α joining a to b in $\bar{\Sigma}$ which is a concatenation of subpaths contained in a leaf or in $\bar{D}$. Let t_0 be the last instant for which $\alpha(t_0)$ maps to $d(0)$ in $T_{\bar{\Sigma}}$. Then for η small enough, the restriction of α to $[t_0, t_0+\eta]$ is a desired lift. $\square$

5 Complexity of exotic components.

Proposition 5.1. *Consider $(T, \Gamma, \mathcal{H})$ an almost geometric action of a pair with a nice decomposition as a graph of actions $\mathcal{G}$ (definition 3.5). Then, for each exotic vertex action (T_v, Γ_v^0), there is a simplicial action (T'_v, Γ_v^0) with trivial arc stabilizer, in which every point stabilizer of (T_v, Γ_v^0) fix a point, and satisfying*

$$C(T_v) \leq C(T'_v).$$

Furthermore, there is $\mathcal{G}'$ be a graph of actions obtained from $\mathcal{G}$ by changing T_v to T'_v such that $(T_{\mathcal{G}'}, \Gamma, \mathcal{H})$ is an almost geometric action with no exotic component, its arc stabilizers are in $\mathcal{C}$ if those of T are, and we have

$$C(T) \leq C(T_{\mathcal{G}'}) + 2\gamma.$$

Remark. We actually construct T'_v as an approximation of T_v.

Lemma 5.2. *Let Σ be the suspension of a pure exotic system of isometries, and let $\{\mathcal{L}_1, \dots, \mathcal{L}_n\}$ be a finite set of leaves of Σ. Then there exists a foliated 2-complex Σ' whose leaves are compact, a finite set $\{\mathcal{L}'_1, \dots, \mathcal{L}'_n\}$ of leaves of Σ' and a map $f : \Sigma' \to \Sigma$ with the following properties.*

- *(i) f is a homotopy equivalence, $f(D') \subset (D)$, f is an isometry in restriction to each component of D', and f sends any leaf of Σ' to a subset of a leaf of Σ.*
- *(ii) $\forall i = 1, \dots, n$, $f(\mathcal{L}'_i) \subset \mathcal{L}_i$ and $f_* : \pi_1(\mathcal{L}'_i) \to \pi_1(\mathcal{L}_i)$ is onto*
- *(iii) $C(\Sigma) \leq C(\Sigma')$*

Proof of the Proposition using the Lemma. Consider a nice decomposition $(T_{\mathcal{G}}, \Gamma)$ of T. For each vertex v corresponding to a pure exotic system of isometries Σ_v, let Σ'_v be given by the lemma, where $\{\mathcal{L}_1, \dots, \mathcal{L}_n\}$ is the set of leaves corresponding to attaching points of edges incident on v in the graph of actions. Let $T_v = T_{\bar{\Sigma}_v}$ and $T'_v = T_{\bar{\Sigma}'_v}$. f

induces an equivariant morphism of $\mathbb{R}$-trees between (T_v, Γ_v^0) and (T'_v, Γ_v^0) showing that edge stabilizers of (T'_v, Γ_v^0) are trivial. Property (ii) says that stabilizers of attaching points of T_v fix a point in T'_v.

Let $\mathcal{G}'$ be a graph of actions obtained by replacing T_v by T'_v for each exotic vertex v. In view of corollary 2.3, we may move attaching points to get $C(T) \leq C(T_{\mathcal{G}'}) + 2\gamma$. $\square$

Proof of the Lemma. Following [Gui98], because generators are independent, any loop in a leaf is either nullhomotopic or contains a singular edge, i. e. one of the two sides of a band which is contained in a leaf. Thus, any non simply-connected leaf $\mathcal{L}$ is one of the finitely many singular leaves, and it contains a finite graph $K(\mathcal{L})$ whose fundamental group generates $\pi_1(l)$ (if $\mathcal{L}$ is simply connected, we set $K(\mathcal{L})$ to one point).

We will narrow a band by a small amount δ as in [Gui98, BFb]: given an edge $e = \{b\} \times [0,1]$ in the boundary of a band $[a,b] \times [0,1]$, narrowing this band at e means replacing it by $[a, b-\delta] \times [0,1]$. The obtained foliated 2-complex is denoted by Σ_δ and $\mathcal{L}_e$ denotes the Σ-leaf containing e. The condition (i) for inclusion $\Sigma_\delta \subset \Sigma$ requires that the map induced by inclusion $\mathcal{L}_e \setminus e \hookrightarrow \mathcal{L}_e$ on fundamental groups is onto. During iteration of the pruning process on Σ, the number of singular edges keeps growing but the number of singular leaves stays bounded, and for every non simply-connected leaf, $K(\mathcal{L})$ remains unchanged. Therefore, after sufficiently long iteration of the pruning process, there is a boundary of a band e which does not belong to $K(\mathcal{L})$ for any singular leaf $\mathcal{L}$.

If δ is small enough, the part of the band removed $(b-\delta, b] \times (0,1)$ does not meet $K(\mathcal{L})$ for any non simply-connected leaf $\mathcal{L}$. Thus, properties (i) and (ii) hold if we choose for $\mathcal{L}'_i$ the Σ_δ-leaf containing $K(\mathcal{L}_i)$.

To handle property (iii), we take extra care in the choice of e and δ (but no extra pruning will be necessary). Given a singular leaf $\mathcal{L}$, consider the minimal connected graph $K'(\mathcal{L}) \subset \mathcal{L}$ containing $K(\mathcal{L})$ and every singular edge in $\mathcal{L}$. We choose e so that all singular edges of $\mathcal{L}_e \setminus e$ are on the same component of $K'(\mathcal{L}_e) \setminus e$. We denote by $\mathcal{L}'_e$ the Σ_δ-leaf containing all the singular edges of $K'(\mathcal{L}_e) \setminus e$. We choose δ small enough so that for every singular leaf $\mathcal{L} \neq \mathcal{L}_e$, $K'(\mathcal{L}) \subset \Sigma_\delta$, and $K'(\mathcal{L}_e) \setminus e \subset \Sigma_\delta$. Moreover, we choose δ so that the new boundary $e_\delta = \{b-\delta\} \times [0,1]$ of the narrowed band is in a singular leaf of Σ (this still enables us to choose δ as small as we want since the union of singular leaves is dense in Σ [Gab96]). We denote by $\mathcal{L}_{e_\delta}$ the Σ_δ-leaf containing e_δ.

Consider a singular leaf $\mathcal{L} \neq \mathcal{L}_e$, and let $\mathcal{L}'$ be the Σ_δ-leaf containing $K'(\mathcal{L})$. Recall that $\mathcal{DL}$ is the graph of directions of $\mathcal{L}$ and that $q : \mathcal{DL} \to \mathcal{L}$ is the natural projection. Every connected component of $\mathcal{DL}$ contains a point whose projection in $\mathcal{L}$ is the endpoint of a singular edge. Thus, $q^{-1}(K'(\mathcal{L}))$ meets every connected component of $\mathcal{DL}$. Therefore, if $\mathcal{L} \neq \mathcal{L}_e$, then $C_{\Sigma_\delta}(\mathcal{L}') \geq C_\Sigma(\mathcal{L})$.

Similarly, one has $C(\mathcal{L}'_e) \geq C(\mathcal{L}_e) - 1$: this is because the preimage in $\mathcal{DL}_e$ of the component of $K'(\mathcal{L}'_e) \setminus e$ containing all the singular edges distinct from e meets all the connected components of $\mathcal{DL}$ but one. This gives the inequality $C(\Sigma_\delta) \geq C(\Sigma) - 1$.

Now remember that we chose δ so that e_δ lies in a singular leaf $\mathcal{L}$ of Σ, and look at the leaf $\mathcal{L}_{e_\delta}$. If $\mathcal{L} \neq \mathcal{L}_e$, then we get $C_{\Sigma_\delta}(\mathcal{L}_{e_\delta}) \geq C_\Sigma(\mathcal{L}) + 1$ because the new singular edge e_δ disconnects a component of $\mathcal{DL}$. Similarly, if $\mathcal{L} = \mathcal{L}_e$, we get $C_{\Sigma_\delta}(\mathcal{L}_{e_\delta}) \geq C_\Sigma(\mathcal{L}_e)$. Thus, on the whole, $C(\Sigma_\delta) \geq C(\Sigma)$ and (iii) holds.

It is not quite true that Σ_δ has compact leaves but it is proved in [Gui98] that the only minimal components that may appear in Σ_δ are exotic and that performing such a band narrowing decreases the number of ends of singular leaves. Therefore, after a finite sequence of pruning and narrowing operations, we get a foliated 2-complex Σ' with compact leaves (see [Gui98]). This proves the lemma. $\square$

6 Complexity of surface components

Proposition 6.1. *Consider $(T,\Gamma,\mathcal{H})$ an almost geometric action of a pair with a nice decomposition as a graph of actions $\mathcal{G}$ (definition 3.5).*

Then, for each surface type vertex action (T_v,Γ_v^0), there is a simplicial action (T_v',Γ_v^0) with cyclic arc stabilizer, in which every point stabilizer of (T_v,Γ_v^0) fix a point, and satisfying

$$C(T_v)\leq 3C(T_v')+6.$$

There is a graph of actions $\mathcal{G}'$ obtained from $\mathcal{G}$ by changing T_v to T_v' such that $(T_{\mathcal{G}'},\Gamma,\mathcal{H})$ is an almost geometric action with no exotic or surface component, arc stabilizers are in $\mathcal{C}$ if those of T are, and satisfying

$$C(T)\leq C(T_{\mathcal{G}'})+10\gamma.$$

Proof. Unlike in the exotic case, we have now that for every length-0 edge e, its edge group Γ_e in $\mathcal{G}$ is in $\mathcal{C}$. This is because point stabilizers in a surface-type or homogeneous action (T_v,Γ_v^0) are cyclic (at most $\mathbb{Z}/2$ in the axial case) and $\mathcal{C}$ is closed under cyclic extension. Therefore, $\mathcal{G}$ has at most γ reduced vertices and hence at most γ surface-type and homogeneous components.

Let v be a vertex such that T_v is of surface type, let Σ_v be the corresponding foliated surface, and $\rho_v:\pi_1(\Sigma_v)\to\Gamma_v^0$ defining the cover of Σ_v. Γ_0 is the fundamental group of the orbifold with boundary obtained from Σ_v by collapsing a boundary component B to a conic point of angle $2\pi/n$ where n is the index of $\pi_1(B)\cap\ker\rho_v$ in $\pi_1(B)$ (when $n=\infty$, don't collapse B). We rather consider the surface Σ_v' obtained from Σ_v by collapsing to a (regular) point each boundary components with $n=1$. Each interior singularity s of the foliation induced on Σ_v' corresponds to such a collapse. Because of the tameness, it is a $p(s)$-pronged singularity with $p(s)\geq 2$. When $p(s)=2$, it is a *false* singularity. For boundary singularities, $p(s)$ will denote the number of branches going into the interior of Σ_v so that $p(s)=0$ for regular points on the boundary (see [FLP79] for background on measured foliations on surfaces).

Now let's compute $C(T_v)$ using lemma 4.1. A singular leaf $\mathcal{L}$ in Σ_v is composed of a circle made of k singular edges, and of a semiline made of regular edges coming out of each vertex which does not lie in ∂D. Thus $\mathcal{DL}$ has $C(\mathcal{L})=k-\#(\mathcal{L}\cap\partial D)$ connected components. Thus, all those complexities add up to $2\#\{\text{bands of }\Sigma_v\}-\#\partial D=-2\chi(\Sigma_v)$. But some of these leaves correspond to non-reduced points: this occurs for each singular leaf $\mathcal{L}$ with complexity 2 such that $\pi_1(\mathcal{L})\subset\ker\rho_v$, in other words for false interior singularities of Σ_v'. Hence,

$$\begin{aligned}C(T_v)&=-2\chi(\Sigma_v)-2\#\{\text{false interior singularities of }\Sigma_v'\}\\&=-2\chi(\Sigma_v')+2\#\{\text{true interior singularities of }\Sigma_v'\}\end{aligned}$$

On the other hand, a standard Euler characteristic argument shows that

$$\begin{aligned}-2\chi(\Sigma_v')&=\sum_{\substack{s\text{ interior}\\\text{singularity of }\Sigma_v'}}[p(s)-2]+\sum_{\substack{s\text{ boundary}\\\text{singularity of }\Sigma_v'}}p(s)\\&\geq\#\{\text{true interior singularities of }\Sigma_v'\}\end{aligned}$$

Thus, $C(T_v) \leq -6\chi(\Sigma'_v)$.

Consider on Σ'_v a maximal system of disjoint, one-sided, simple closed curves. Cutting Σ'_v along those curves gives an orientable surface with same Euler characteristic as Σ'_v. Decompose it into $-\chi(\Sigma'_v)$ pants. The boundaries of these pants give at least $-\chi(\Sigma'_v)-1$ not boundary-parallel disjoint curves in Σ'_v. Each vertex of the action (T'_v, Γ^0_v) dual to this system of curves has a non-cyclic stabilizer, so must be reduced. Thus $C(T'_v) \geq -2\chi(\Sigma'_v) - 2$. Therefore, we get the expected inequality $C(T_v) \leq 3C(T'_v) + 6$.

Let $\mathcal{G}'$ be the graph of actions obtained by replacing the vertex tree T_v by T'_v for each surface type vertex action T_v (this is possible because fundamental groups of boundary components of Σ'_v fix a point in T'_v). In view of corollary 2.3, we get

$$\begin{aligned} C(T) &\leq C(T_{\mathcal{G}'}) + \sum_{T_v \text{ surface type}} \Big(C(T_v) - C(T'_v) \Big) + 2\gamma \\ &\leq C(T_{\mathcal{G}'}) + \sum_{T_v \text{ surface type}} \Big(2C(T'_v) + 6 \Big) + 2\gamma \end{aligned}$$

If we replace homogeneous actions of $\mathcal{G}'$ by trivial actions, the obtained simplicial action of Γ has complexity at least $\sum_{T_v \text{ surface type}} C(T'_v)$ (use lemma 2.2 where one can take $a = 0$ since no vertex of T'_v has a cyclic stabilizer). Thus $\sum_{T_v \text{ surface type}} C(T'_v) \leq \gamma$. Since the number of surface type vertices is at most γ, we get $C(T) \leq C(T_{\mathcal{G}'}) + 10\gamma$. □

7 Complexity of homogeneous components

Proposition 7.1. *Let $(T, \Gamma, \mathcal{H})$ be an almost geometric action of a pair having a nice decomposition as a graph of actions $\mathcal{G}$ with no exotic or surface component. Assume that arc stabilizers of T are in $\mathcal{C}$.*

Then $C(T) \leq 3\gamma + \gamma 2^{\gamma + \dim H_1(\Gamma;\mathbb{Z}/2) - 1}$

Proof. Like in proposition 6.1, for every edge e of $\mathcal{G}$, its edge group Γ_e in $\mathcal{G}$ is in $\mathcal{C}$ (even if e has length 0). Therefore, the number of homogeneous components in $\mathcal{G}$ is bounded by γ. If we have a bound B for the complexity of a homogeneous component occurring in $\mathcal{G}$, we deduce from lemma 2.2 that

$$C(T) \leq C(\tilde{\mathcal{G}}_0) + 2a + \sum_v C(T_v) \leq 3\gamma + \gamma B.$$

Let v be a vertex corresponding to a homogeneous component in $\mathcal{G}$. Since T_v is a line, $C(T_v) = 0$ when v is an orientable homogeneous component. When v is non-orientable, then $\Gamma^0_v \simeq \mathbb{D}_n$ acts faithfully on $\mathbb{R}$ as a subgroup of Isom $(\mathbb{R})$ generated by a reflexion and n rationally independent translations. Since T_v has complexity 2^n, we just have to bound n.

Let $\mathcal{G}'$ be the graph of groups obtained from the graph of groups underlying $\mathcal{G}$ by killing edge groups and let $\Gamma' = \pi_1(\mathcal{G}')$. Γ maps onto Γ' which maps onto Γ'_v (Γ'_v is the vertex group of v in $\mathcal{G}'$). Let C be the subspace of $H_1(\Gamma^0_v;\mathbb{Z}/2)$ generated by the images of the edge groups of edges incident on v. Note that $\dim C \leq \gamma$ since images of edge groups in Γ^0_v are cyclic. We have $H_1(\Gamma'_v;\mathbb{Z}/2) = H_1(\Gamma^0_v;\mathbb{Z}/2)/C$ and

$$n + 1 = \dim H_1(\Gamma^0_v;\mathbb{Z}/2) = \dim H_1(\Gamma'_v;\mathbb{Z}/2) + \dim C \leq \dim H_1(\Gamma;\mathbb{Z}/2) + \gamma.$$

Therefore, $C(T_v)$ is bounded by $B = 2^{\dim H_1(\Gamma;\mathbb{Z}/2)+\gamma-1}$. □

8 Conclusion

Here, we put everything together to prove the Theorem. Consider a finitely presented pair $(\Gamma, \mathcal{H})$. Let $\mathcal{C}$ be a class of subgroups of Γ stable by taking subgroups and cyclic extensions such that $(\Gamma, \mathcal{H})$ satisfies the accessibility condition with respect to $\mathcal{C}$ with accessibility constant γ. Consider a stable action $(T, \Gamma, \mathcal{H})$ with arc stabilizers in $\mathcal{C}$.

By lemma 3.4, $(T, \Gamma, \mathcal{H})$ is a strong limit of almost geometric actions $(T_i, \Gamma, \mathcal{H})$ having a nice decomposition. By Proposition 3.1, $C'(T) \leq \liminf C(T_i)$.

We now fix an index i and we bound $C(T_i)$. Propositions 5.1 and 6.1 provide from T_i an action $(T'_i, \Gamma, \mathcal{H})$ without exotic components or surface components with arc stabilizers in $\mathcal{C}$ such that $C(T_i) \leq C(T'_i) + 12\gamma$. Proposition 7.1 concludes that $C(T'_i) \leq 3\gamma + 2^{\gamma + \dim H_1(\Gamma;\mathbb{Z}/2) - 1}$. $\square$

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