Theorem (Sela). *Consider a minimal action of a finitely generated group* Γ *on an* $\mathbb{R}$*-tree* T*.*
Hypotheses:

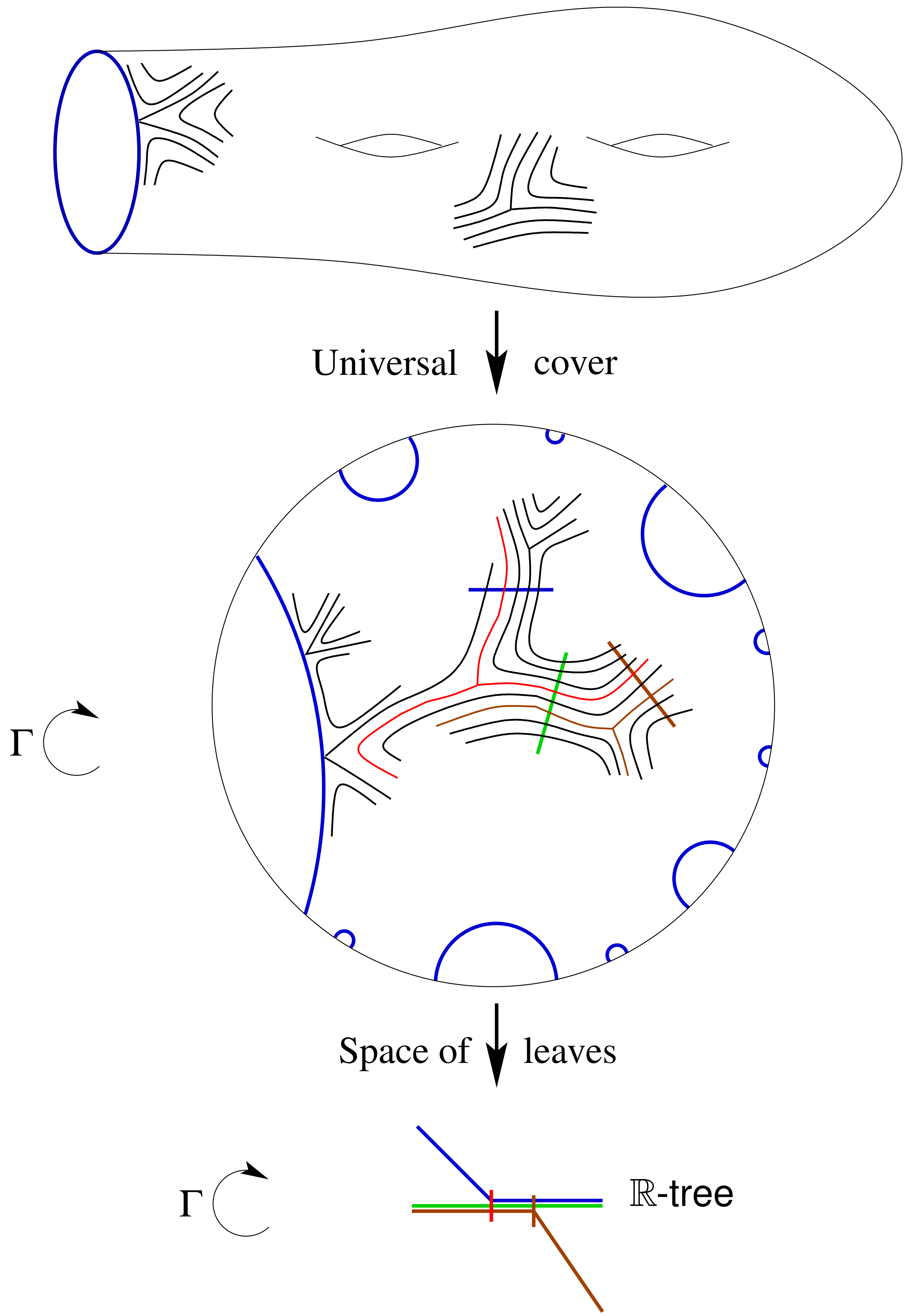
- Γ *is freely indecomposable*
- *tripod stabilizers are trivial*
- super-stability:

$$\text{if } \mathtt{Stab}\, I \neq \{1\} \text{ and } J \subset I, \text{ then } \mathtt{Stab}\, J = \mathtt{Stab}\, I$$

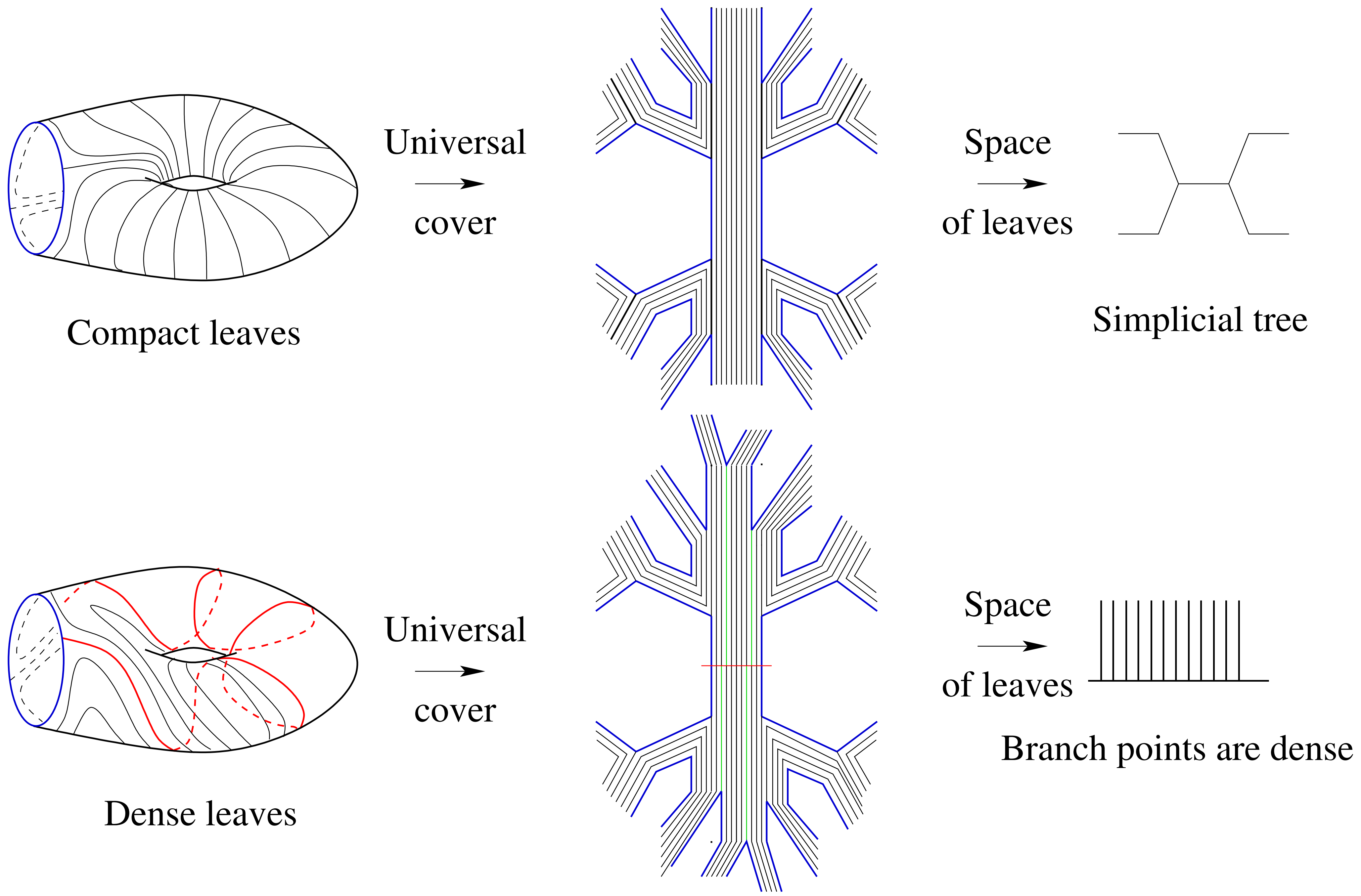
Conclusion: T *splits as a graph of actions on* $\mathbb{R}$*-trees, vertex actions being*

- *either the action of a finitely generated group on a line (may have a kernel)*
- *or the action dual to a measured foliation on a* 2*-orbifold with boundary (arc stabilizers are trivial).*

In particular, if T *is not a line, then* Γ *splits over a cyclic group or a cyclic extension of an arc stabilizer.*

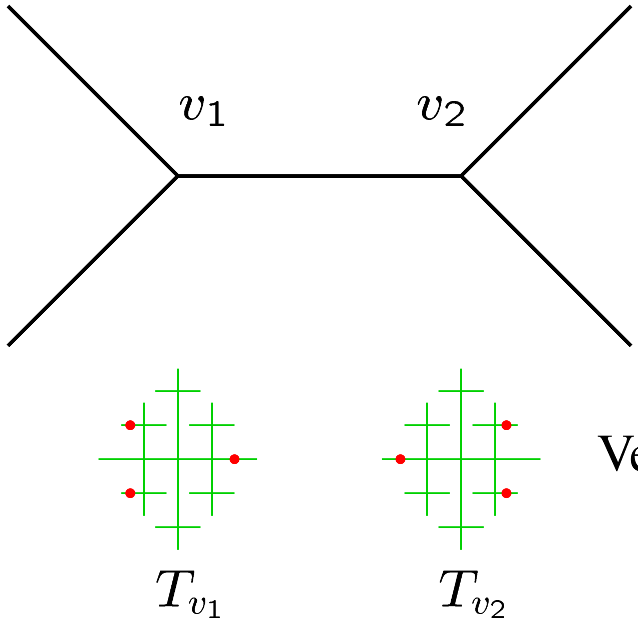


Universal
cover
Γ
Space of
leaves
Γ
ℝ-tree


Universal
cover
Space
of leaves
Compact leaves
Simplicial tree
Universal
cover
Space
of leaves
Dense leaves
Branch points are dense

Graph of actions

Data defining a graph of actions:

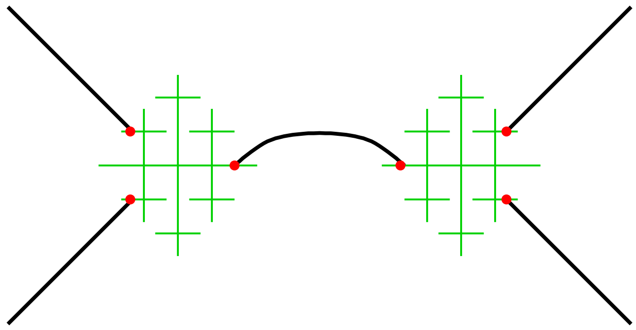


Simplicial skeleton $S \circlearrowleft \Gamma$

+ lengths for edges

Vertex actions + attaching points

Resulting $\mathbb{R}$-tree:



Criterion for splitting as a graph of actions. T is *nicely* covered by an equivariant family Y_i of closed subtrees:

- every arc is covered by finitely many Y_i
- if $Y_i \cap Y_j$ contains an arc, then $Y_i = Y_j$

Strategy of the proof

Step 1. Build a sequence of foliated 2-complexes Σ_k from T

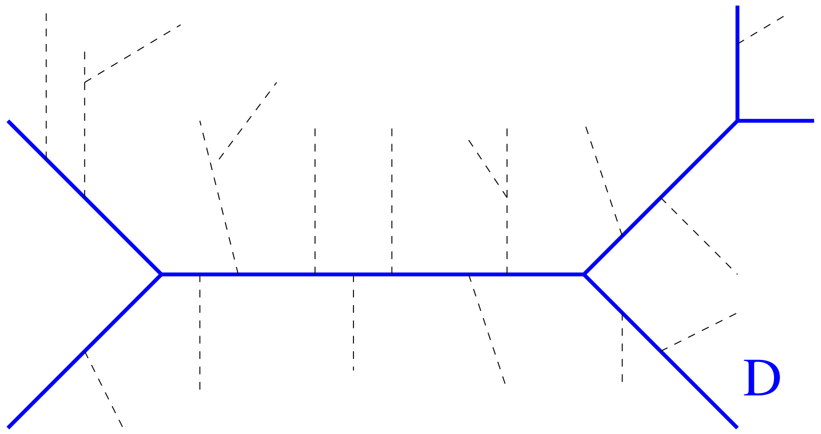
Step 2. Build $\mathbb{R}$-trees T_k from the 2-complexes

Step 3. Dynamical decomposition of Σ_k gives a decomposition of T_k, each vertex action being dual to a foliated 2-complex with dense leaves

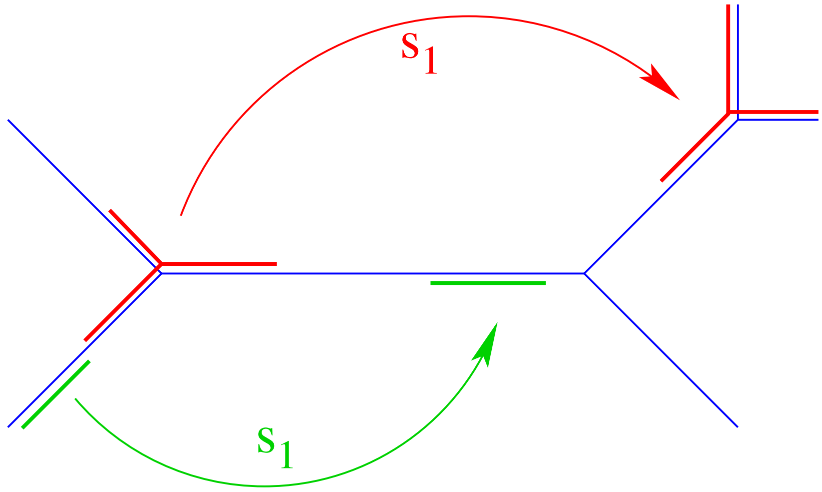
Step 4. The decomposition of T_k maps nicely into $T \rightsquigarrow$ decomposition of T

Step 1. From the $\mathbb{R}$-tree to foliated 2 complexes

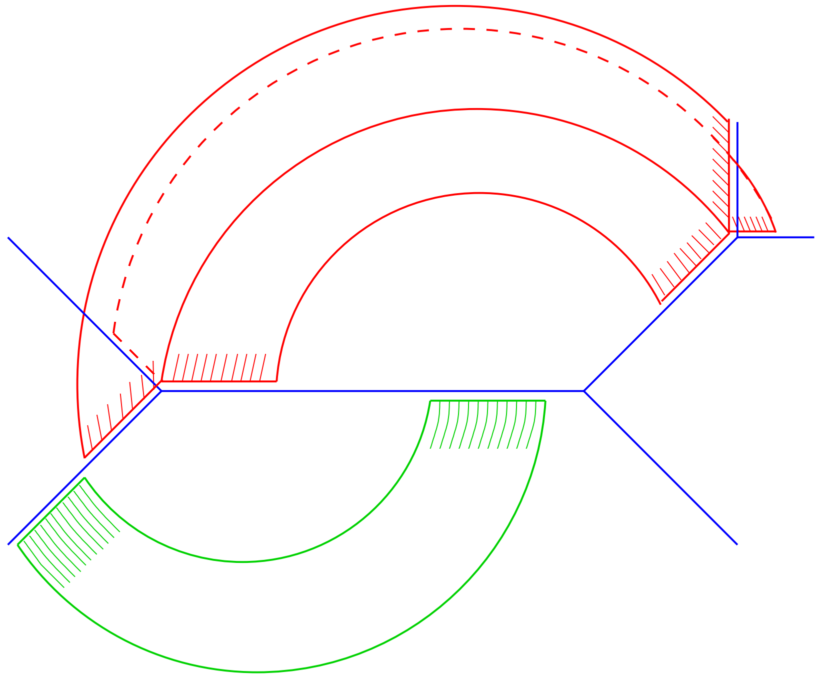
Let $\Gamma = \langle s_1, \ldots, s_n | r_1, \ldots, r_p, \ldots \rangle$.



finite subtree $D \subset T$



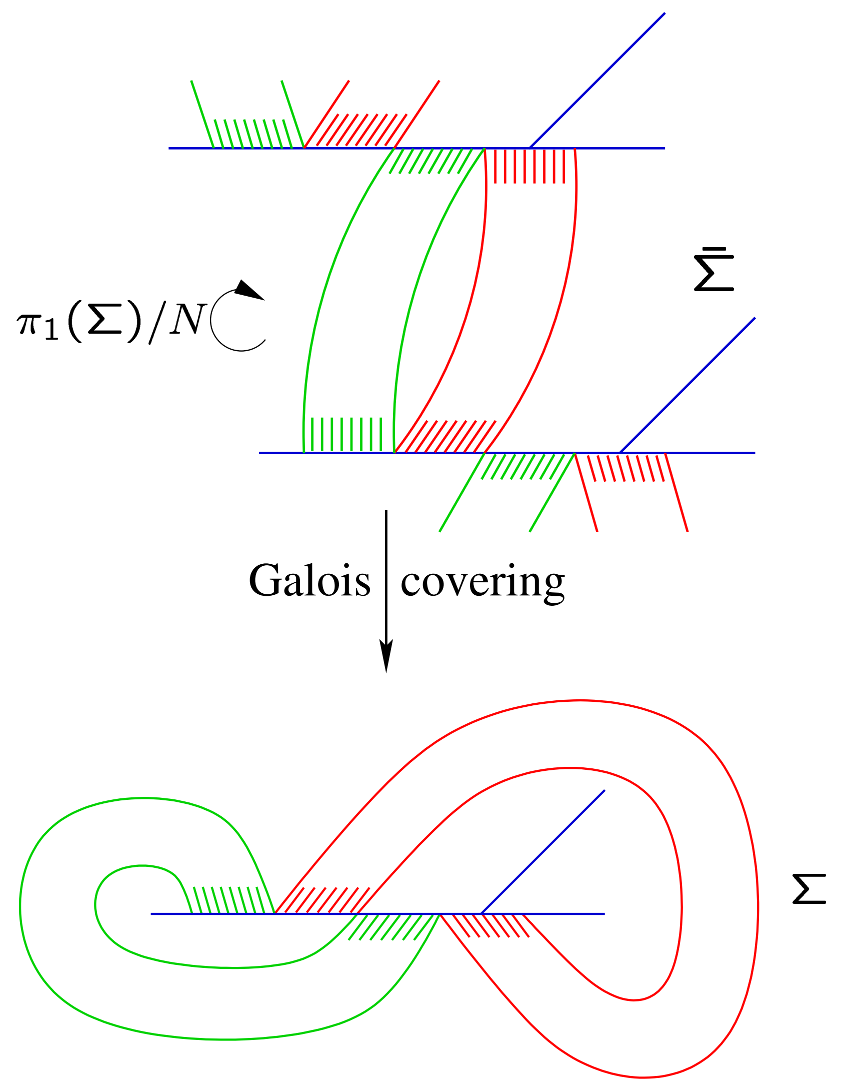
Maximal restriction of the generators $D \to D$



Suspension

Foliated 2-complex Σ

Step 2. From foliated 2-complexes to $\mathbb{R}$-trees



$$\pi_1(\Sigma) = \langle s_1 \ldots, s_n | \emptyset \rangle$$

$\longrightarrow$ Leaf space made Hausdorff:
$\mathbb{R}$-tree

Theorem (LP): if N is normally generated by loops contained in curves, then the leaf space of $\bar{\Sigma}$ is an $\mathbb{R}$-tree

Sequence of actions

Exhausting sequence of finite trees D_k

$$\rotatebox{-90}{$\rightsquigarrow$}$$

Foliated 2-complexes Σ_k

$$\rotatebox{-90}{$\rightsquigarrow$}$$

$\mathbb{R}$-trees $T_k \circlearrowleft \Gamma_k = \langle s_1, \ldots, s_n | \text{more and more relations} \rangle$

$$\begin{array}{ccccccc}
T_k & \longrightarrow & T_{k+1} & \longrightarrow & \cdots & \longrightarrow & T \\
\circlearrowleft & & \circlearrowleft & & & & \circlearrowleft \\
\Gamma_k & \twoheadrightarrow & \Gamma_{k+1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & \Gamma
\end{array}$$

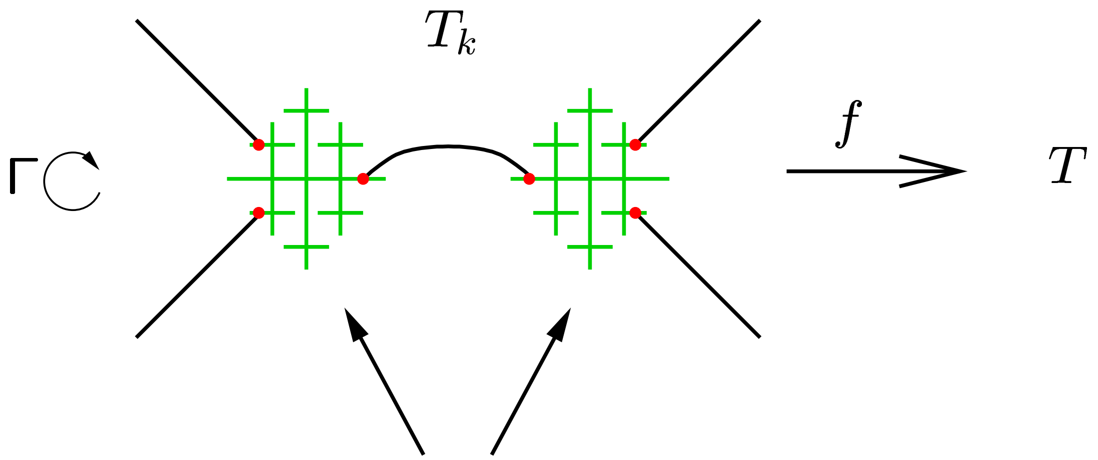
(maps between trees are morphisms of $\mathbb{R}$-trees)

Step 3. Dynamical decomposition of Σ_k

Σ_k splits into finitely many pieces:

- minimal components (every leaf is dense)
- simplicial components (every leaf is compact)

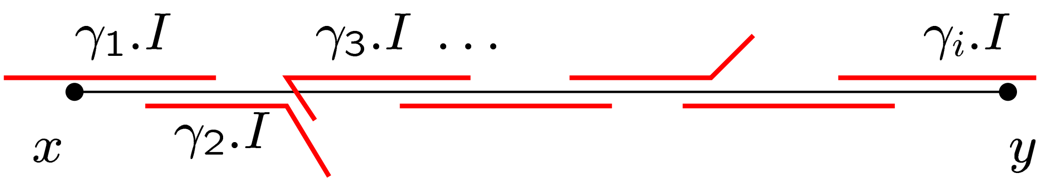
T_k splits as a graph of actions,
vertex actions are dual to minimal components.



Vertex actions dual to minimal components

Mixing property:

Def: $Y \subset T$ is mixing if $\forall I \subset Y$, $\forall x, y \in Y$,
there exists a chain of translates of I joining x to y.



Properties.

1. Vertex actions corresponding to minimal components are mixing.
2. If $Y \subset T_k$ is mixing, then $f(Y) \subset T$ is mixing.
3. **mixing + stability:**

$$\gamma \text{ fixes an arc in } Y \Rightarrow \gamma \text{ fixes } Y.$$

4. **mixing + stability + trivial tripod stabilizers:**

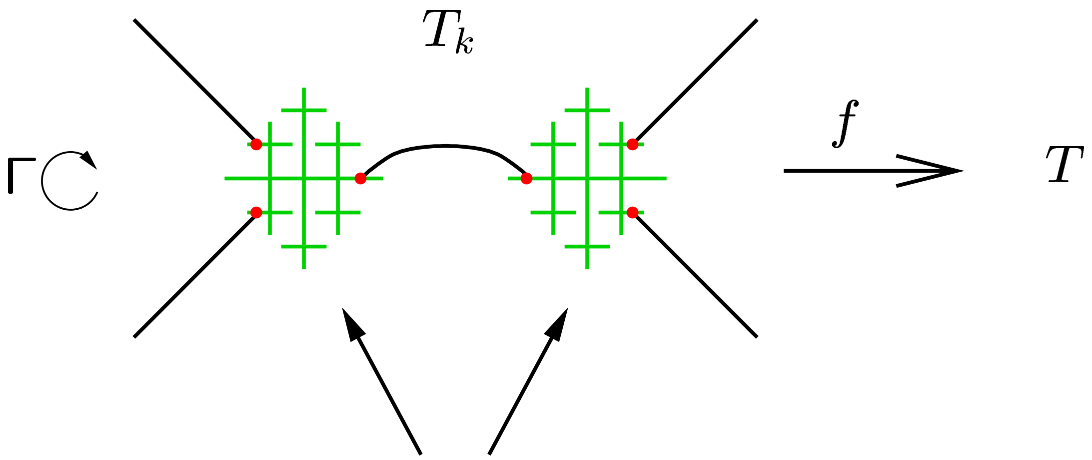
$$\gamma \neq 1 \text{ fixes an arc in } Y \Rightarrow Y \text{ is a line.}$$

Using stability of T, the following can be assumed in T_k:

$$\gamma \text{ fixes an arc in } T_v \Rightarrow \gamma \text{ fixes } T_v.$$

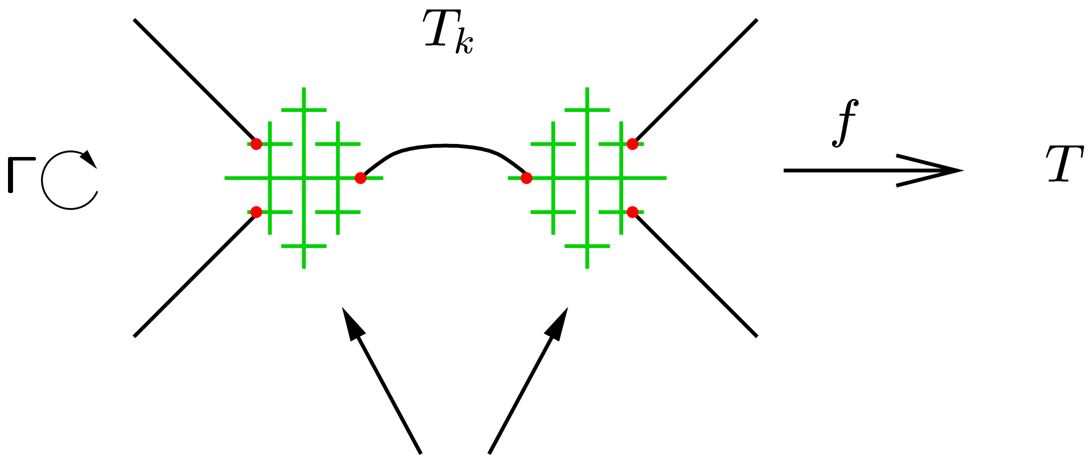
by-product: point stabilizers of T_k inject into Γ.

Summary



Vertex actions dual to minimal components

- vertex actions are mixing
- if γ fixes an arc in $T_v \Rightarrow \gamma$ fixes T_v
- point stabilizers inject in Γ



Vertex actions dual to minimal components

Idea:
If $Y \subset T$ is a mixing subtree $\neq$ line,
and I is an arc with $\mathtt{Stab}\ I \neq \{1\}$
then $Y \cap I$ is at most one point.

Def: $E_1 = \{$maximal arcs with $\mathtt{Stab}\ \neq 1$
$E_2 = \{f(T_v) | T_v$ vertex action in T_k with trivial arc stabs$\}$

Goal: Prove that $E_1 \cup E_2$ is a nice covering of T:

- every arc in T is covered by finitely many $Y_i \in E_1 \cup E_2$
- if $Y \cap Y'$ contains an arc, $(Y, Y' \in E_1 \cup E_2)$, then $Y = Y'$
- $f(T_v)$ is either a line or a surface-type action

Finding arc stabilizers

Scott Lemma.
A sequence of non-trivial free splittings $\Gamma_k \circlearrowright S_k$ with equivariant maps $S_k \to S_{k+1}$ provides a non-trivial free splitting of Γ.

Corollary. For k large enough, every edge of positive length in T_k has a non-trivial stabilizer.

Consequence in T: Any arc $I \subset T$ is covered by finitely many Y_i.

Classification of minimal components in Σ_k

- **Homogeneous components:** extension property
- **Surface type components:** T_v is dual to a measured foliation on a 2-orbifold
- **Exotic components:** otherwise (well understood).

Properties:

- T_v dual to a homogeneous component $\Rightarrow T_v$ is a line
- Stabilization of non-homogeneous components

Corollaries:

- T_v embed into T
- If $f(T_{v_1}) \cap f(T_{v_2})$ contains an arc, then $f(T_{v_1}) = f(T_{v_2})$.